Intro to Linear Algebra

Note Title 2016-05-16

Review

1. $2x + 3y = -8$
2. $-3x - y = -9$

$2(5) + 3(-6) = -8$
$-3(5) - (-6) = -9$

Solve by substitution

from 2: $-3x + 9 = y$ (3)

Subst into 1: $2x + 3(-3x + 9) = -8$

$2x - 9x + 27 = -8$

$-7x = -35$

$x = 5$

Subst into 3: $-3(5) + 9 = y$

$-6 = y$

Solution: $x = 5$, $y = -6$

Solve by elimination

multiply 2 by 3: $-9x - 3y = -27$

add to 1: $2x + 3y = -8$

$-7x = -35$

$x = 5$

Subst into 2: $-3(5) - y = -9$

$-6 = y$

Solution: $x = 5$, $y = -6$

New: Using an augmented matrix, we use only the coefficients and do Gaussian Elimination into reduced row echelon form.

$$\begin{aligned} 2x + 3y &= -8 \\ -3x - y &= -9 \end{aligned} \Rightarrow \begin{array}{ccc} x & y & c \\ \end{array} \left[\begin{array}{cc|c} 2 & 3 & -8 \\ -3 & -1 & -9 \end{array}\right]$$

$$\begin{array}{c} 3R_1 \\ 2R_2 \end{array} \left[\begin{array}{cc|c} 2 & 3 & -8 \\ -3 & -1 & -9 \end{array}\right]$$

$$\begin{array}{c} \\ R_1 + R_2 \end{array} \left[\begin{array}{cc|c} 6 & 9 & -24 \\ -6 & -2 & -18 \end{array}\right]$$

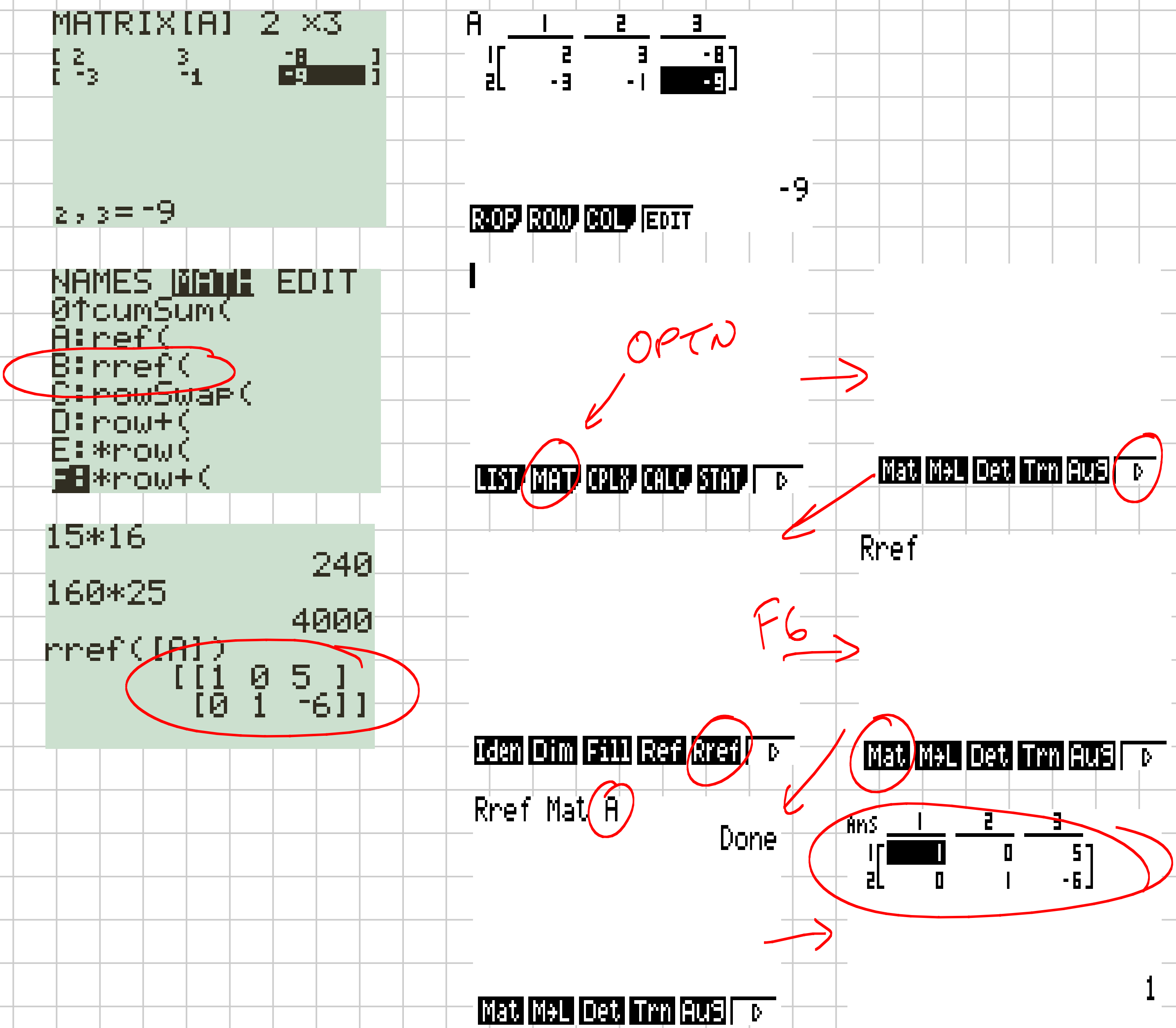
$$R_2/7\quad \left[\begin{array}{cc|c} 6 & 9 & -24 \\ 0 & 7 & -42 \end{array}\right]$$

$$R_1 - 9R_2\quad \left[\begin{array}{cc|c} 6 & 9 & -24 \\ 0 & 1 & -6 \end{array}\right]$$

$$R_1/6\quad \left[\begin{array}{cc|c} 6 & 0 & 30 \\ 0 & 1 & -6 \end{array}\right]$$

$$\begin{array}{ccc} x & y & c \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -6 \end{array}\right] \Rightarrow \begin{array}{l} x = 5 \\ y = -6 \end{array}$$

This looks longer, however, we can get the calculator to solve this!



Solve
$$\begin{aligned} 3x - 2y + z &= -5 \\ -2x + 4y - 2z &= 18 \\ 4x - 6y - 3z &= -7 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & -2 & 1 & -5 \\ -2 & 4 & -2 & 18 \\ 4 & -6 & -3 & -7 \end{array}\right] \overset{\text{Rref}}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Solve
$$\begin{aligned} 3x - 2y + z &= -20 \\ -2x + 4y - 2z &= 28 \\ 4x - 6y - 3z &= -39 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & -2 & 1 & -20 \\ -2 & 4 & -2 & 28 \\ 4 & -6 & -3 & -39 \end{array}\right] \overset{\text{Rref}}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Notice both systems have the same system except for the constant terms. We can do a bunch of solutions by finding the inverse of the matrix. (It's an expensive computation, so we want to use this more than once.)

$$A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 4 & -6 & -3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 7/24 & 13/48 & -1/12 \\ 1/12 & -5/24 & -1/6 \end{bmatrix}$$
$$= \frac{1}{48}\begin{bmatrix} 24 & 12 & 0 \\ 14 & 13 & -4 \\ 4 & -10 & -8 \end{bmatrix}$$

$$C = \begin{bmatrix} -5 \\ 18 \\ -7 \end{bmatrix}$$

We only need to plug in new C's to solve more similar systems.

Let's label the matrices' elements:

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$

To find the solution, we multiply $A^{-1} \cdot C$. But matrix multiplication is different from numerical multiplication.

$$A^{-1} \cdot C = \begin{bmatrix} A_{11} C_{11} + A_{12} C_{21} + A_{13} C_{31} \\ A_{21} C_{11} + A_{22} C_{21} + A_{23} C_{31} \\ A_{31} C_{11} + A_{32} C_{21} + A_{33} C_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$

On computers and calculators, it's much faster than doing Gaussian Elimination.

What are other uses? Computer Graphics! Recall that order is important when applying multiple transformations. Matrix multiplication is not commutative!

Transformation Matrices

Transformation matrix is a basic tool for transformation. A matrix with n x m dimensions is multiplied with the coordinate of objects. Usually 3 x 3 or 4 x 4 matrices are used for transformation. For example, consider the following matrix for various operation.

$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$	$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$Sh = \begin{bmatrix} 1 & sh_x^y & sh_x^z & 0 \\ sh_y^x & 1 & sh_y^z & 0 \\ sh_z^x & sh_z^y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Translation Matrix	**Scaling Matrix**	**Shear Matrix**
$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & -sin\theta & 0 \\ 0 & sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\theta & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
	Rotation Matrix	

TutorialsPoint.com

To do 3D Graphics, we actually need a 4D matrix - this is to handle translations.