

MATH 380 – 1081 – Term Paper

William Rowan Hamilton's Road to Quaternions

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Student number

April 7, 2008

Submitted to: Dr. Archibald

INTRODUCTION

William Rowan Hamilton (1805-1865) was born and worked in Ireland. Being a relatively modern mathematician from an intellectual family, his life and work was fairly well documented. This naturally leads to the topic of this paper which is what motivated Hamilton to discover quaternions and what quaternions are. His road to quaternions started early as an eager young student who read many books from many different authors and in many different languages. After completing distinguished works in light and optics, Hamilton came up with an algebra for complex numbers or as he termed *number couples*. This was followed by an unsuccessful search for an algebra for *triplets* which was finally abandoned when he discovered quaternion algebra was the solution to the problem. This stems from Hamilton's willingness to abandon the commutative law for multiplication and therefore one of the rules thought to be required for an algebra. Thereafter, writing about quaternions would consume the majority of Hamilton's time for the rest of his life. Quaternions had very little practical impact, so the emphasis of this paper leans more to the thinking behind them and how it impacted the thinking of other mathematicians.

QUATERNIONS

Firstly, quaternion algebra is the first algebra to abandon a rule of algebra, specifically, the *commutative law for multiplication*. Prior to this, any algebra was thought to require all the rules of ordinary algebra: addition, subtraction, multiplication and division with identity, associative, distributive and commutative laws. Quaternions have four ordered real elements in the form:

$$x = a + bi + cj + dk \text{ where } \mathbf{i}, \mathbf{j}, \text{ and } \mathbf{k} \text{ are orthogonal imaginaries.}$$

It resembles a complex number that has been extended. a , b , c , and d can be thought of as four components on mutually orthogonal axes in four dimensions. Because a quaternion essentially represents a four-dimensional quantity, it can be physically interpreted in different ways. Davis and Snyder, "Introduction to Vector Analysis," explain that when a is zero (a *pure quaternion*), b , c , and d are analogous with a three-

dimensional vector, but in doing so is awkward (332-333). Addition and subtraction are defined as one would think:

$$(a_1+b_1i+c_1j+d_1k) + (a_2+b_2i+c_2j+d_2k) = (a_1+a_2) + (b_1+b_2)i + (c_1+c_2)j + (d_1+d_2)k$$

the components are individually summed. Geometrically, one can think of addition and subtraction as translating a vector in space. Multiplication, on the other hand, can be thought of as rotation of a vector in space. Multiplication is defined as:

$$(a_1+b_1i+c_1j+d_1k)(a_2+b_2i+c_2j+d_2k) = (a_1a_2-b_1b_2-c_1c_2-d_1d_2) + (a_1b_2+a_2b_1-c_1d_2+d_1c_2)i + (a_1c_2+a_2c_1-d_1b_2+b_1d_2)j + (a_1d_2+a_2d_1-b_1c_2+c_1b_2)k$$

using the identities as defined by William Hamilton in his “Lectures on Quaternions” (70):

$$i^2 = j^2 = k^2 = ijk = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$$

The last six identities are the same as those used in vector calculus when using a vector product. While the first three are similar to the dot product except that they are equal to **1** instead of **-1**; this is to be expected since **i**, **j** and **k** represent unit basis vectors in vector analysis while quaternions have **i**, **j** and **k** represent the square root of **-1**. There are over 800 pages in Hamilton’s *Lectures* (1853) and certainly cannot be summarized in this paper.

YOUNG HAMILTON

William Hamilton’s road to quaternions started early in his life, because he was recognized as a child prodigy. As such, his early life was documented. Furthermore, Hamilton himself was a prolific writer in terms of books and personal letters. Biographers have volumes of first hand material to base their research. Robert Perceval Graves was a friend of Hamilton and wrote the first biography on Hamilton. Robert’s brothers, John and Charles, were mathematicians and colleagues of William, so Robert has a unique point of view for his biography. Graves’ biography is three volumes of work totalling over 2000 pages. No copies of this were available for this paper.

Sean O’Donnell, in “William Rowan Hamilton: Portrait of a Prodigy”, provides the following details of Hamilton’s early years. At the age of three, he was boarded at the Meath diocesan school in Trim, Ireland. This choice was based in part that Hamilton’s

uncle James (his father's brother) was a cleric at the school. Furthermore, James was William's principal teacher who had a Classics (Arts) honours degree from Trinity College in Dublin (the future workplace of William) and William's father, Archibald, had a desire for William to hold a position in international trade. Most of the accounts of William's achievements are detailed in letters by his aunt Sydney (his father's sister who resided with James and his wife) to his mother Sarah. In the letters, Sydney reported that William was an extremely quick learner who spelt all monosyllabic words in his first year at the school. At the age of five, he had progressed to learning Hebrew, Greek and Latin. By age nine, he read the bible in Hebrew and the *Iliad* in French and developed a habit of carrying his favourite book around to read when it suited him. Sydney also recounted the times when William impressed skeptical adults with his language skills such as with James' superior, Vicar Elliot and James' colleague (also a cleric), Mr. Montgomery. These times were few according to Sydney because James felt that too much attention would hamper his learning. Unfortunately, Sydney died from cancer when William was nine and the letters ended (13-23).

There are accounts of Hamilton having learned thirteen languages by the age of thirteen. O'Donnell researches this claim and concludes that the source is from a letter by his father, Archibald, being more boastful than accurate. O'Donnell's conclusion is based on Hamilton's writings and finds a lack of evidence to support his proficiency of this claim. Although there is evidence that he has some knowledge of these languages (other than English, Hebrew, Greek and Latin), it is not sufficient to prove that he was comfortable with them (23-25). Regardless of the number of languages, it is quite an achievement for someone so young to have learned multiple languages to a level that surprises educators.

According to O'Donnell's research, William was fairly ordinary at mathematics until the age of thirteen. In 1818, he met Zerah Colburn, 14, dubbed "an American calculating prodigy" who was capable of performing arithmetic very quickly and beat Hamilton in a calculating contest. Upon meeting Colburn two years later, Hamilton determined that it was "an usually developed memory" (40-41). This account is

different to that of biographer Thomas Hankins in “Sir William Rowan Hamilton” whose research states that Hamilton asked Colburn for his secrets. Colburn explained his tricks for extracting roots and finding factors. Later, Hamilton examined the methods and discovered that there were limitations (14-15). Regardless of the differences, both biographers agree that this was the period when he displayed a focused interest in mathematics.

TRINITY COLLEGE

Uncle James had been preparing William for studies at Trinity College (Dublin) in the Classics since age three. As stated by Hankins, James included Euclid and Bartholomew Lloyd’s *Analytical Geometry* in William’s lessons because they were required courses at Trinity. At the age of seventeen, 1822, William became ill and arranged to recuperate in Dublin with his cousin Arthur. There he read more mathematics: *Leçons de calcul différentiel* by Jean-Guillaume Garnier and *Mécanique céleste* by Laplace. Hankins claims that he could read these with ease, as evident by finding an error by Laplace (20-21). Hankins further says that William was introduced to a family friend, Charles Boyton, who recently became a Fellow at Trinity for mathematics. Boyton provided Hamilton with undergraduate mathematics books by Woodhouse, Lardner, Lacroix and Newton. Boyton also provided Hamilton books from l’Ecole Polytechnique curriculum by Garnier, Lagrange, Puissant, Francoeur, Monge, Cousin and Malus. Hamilton read all these before entering Trinity in 1823 (23).

SIGNIFICANT WORKS

Hamilton was an extremely prolific writer as supported by the volumes of paper that Robert Graves had to pour through to write his biography on Hamilton. But Hamilton was not only prolific; he was productive. O'Donnell summarizes Hamilton's top seven distinct achievements in chronological order (99).

The first achievement was written while Hamilton was still an undergraduate in 1824 called *On Caustics*. According to O'Donnell, before Hamilton, light properties were

explained with Euclidean geometry. Hamilton's paper described an algebraic method of explaining light properties (99).

Hamilton's second paper was an application of his results from *On Caustics*. O'Donnell states that Hamilton predicted a phenomenon from his algebra on light which is: passing a ray of light through a crystal under controlled conditions produces a hollow circular beam after leaving the crystal. At the time this was a very surprising result (99).

Hamilton's third significant work was written in 1834, *General Methods in Dynamics* which is another paper about light. This paper shows how light can be treated as waves or particles. Treating light as particles is the area of dynamics. O'Donnell claims that no one was able to benefit from this paper until 1926 when Schrödinger used it to explain atomic structure (99). Today, physics students still treat light as particles or waves depending on the model required.

Hamilton's fourth and fifth discoveries are part of this paper which will be described in detail later. These are the *Algebra as the Science of Pure Time* in 1835 and Quaternions in 1843. His sixth achievement was the hodograph which is a different method of explaining planetary orbits written 1846. And finally, his last significant discovery was *Icosian Calculus* in 1856 (100-101). Math students know of Icosian Calculus better as Hamiltonian Cycle and Hamiltonian Path.

COMPLEX NUMBERS

In the *Algebra as the Science of Pure Time*, Hamilton describes his theory of complex number and how imaginary numbers have physical meaning. Hankins recounts how Hamilton came to writing his paper. On April 5, 1825, John Thomas Graves wrote a letter to William that started a discussion about logarithms. The correspondence led to the conjecture by Graves in 1827 that:

[T]he logarithm of a complex number taken to a complex base, two arbitrary integer numbers must be specified to determine the solution, since the complex number of the logarithm and the complex base each introduce an infinite number of possible solutions.

About the same time, Hamilton was thinking about algebra as the science of time in 1827 prior to reading Emmanuel Kant's *Critique of Pure Reason*. After studying Kant, his

thoughts about the geometric interpretations of complex numbers solidified with the following equations:

Let $y = \varphi(x)$, $x = \alpha + \beta i$	using these
$\varphi(\alpha + \beta i) = \varphi_1(\alpha, \beta) + i\varphi_2(\alpha, \beta)$	two real value functions of two real variables
$\frac{\partial \phi_1}{\partial \alpha} = \frac{\partial \phi_2}{\partial \beta}, \frac{\partial \phi_1}{\partial \beta} = -\frac{\partial \phi_2}{\partial \alpha}$	[1] connected by these differential equations
$\gamma_1 = \phi_1(\alpha, \beta), \gamma_2 = \phi_2(\alpha, \beta)$	[2] two new functions of two curve surfaces

The differential equations [1] are called *equations of conjugation* which are known as the “Cauchy-Riemann equations” because Cauchy published them first 1827. However, Hamilton did not read this paper and thus wrote his independently. The two new functions [2] are called *number couples*. Hankins states that Hamilton argued that this algebra is more meaningful than complex numbers because only real numbers are operated on the equations whereas “a complex number written as the sum of a real part and an imaginary part can have no meaning. It expresses the sum of unlike things, and unlike things cannot be added.” Finally, Hankins claims that some mathematics historians believe that this algebra is more significant than quaternions because “[i]t meant breaking away from the real number line as the basis for algebra, and constructing new elements defined in a different way. Hankins closes by saying that this algebra is very close to the definition of an algebraic field (258-266).

Hankins has a long explanation of the relevance of Kant, but O'Donnell summarizes this in a more understandable way, “[h]e formed this notion as a follower of Kant, who believed that space and time were the prime determinants of reality; since geometry was obviously the science of space, Hamilton proposed that algebra would serve a similar role for time” (100). Perhaps Hamilton's training in the Classics made him view mathematics as more of a means than a goal. The theme of space and time would continue with quaternions.

TRIPLETS

Fresh from his discovery, Hamilton wanted to extend his algebra into three-dimensional space which is a natural extension because it would help solve physical

problems in our world. Hankins states that Hamilton worked on and off for over thirteen years searching for triplets. This search was formally written in 1830 in *Triads* (283-284). Clearly, he started thinking about triplets before completing number couples. Hankins explains that Hamilton was attempting to multiply three-dimensional lines but was unsuccessful. And when extending his conjugate functions, he could not find satisfactory partial differential equations. Hankins tells us about an unreliable anecdote in the Hamilton household, where Hamilton's oldest son would ask William every morning during October at breakfast, “Well, Papa, can you multiply triplets?” and William would reply that he could only add and subtract them. Here are the equations that Hamilton was working on:

$$\begin{aligned} (x + iy + jz) & \quad \text{a real and two complex, representing 3 axes in space} \\ i^2 = j^2 = -1 & \quad \text{imaginaries} \\ (x + iy + jz)(x + iy + jz) & = (x^2 - y^2 - z^2) + i(2xy) + j(2xz) + ij(2yz) \\ & \quad \text{squaring of a triplet} \end{aligned}$$

So the problem of squaring a triplet is that the product is not itself a triplet. Hamilton first attempted to eliminate the ij term by setting it to zero. It did not seem reasonable to him, so his next attempt was setting $ij = -ji$ and having the third term equal to

$$yz(ij + ji)$$

but again the algebra did not work out (284, 295-297).

Hamilton had a lot of emphasis on Kant during this time. Perhaps it was his difficulties in solving triplets that caused him to turn to metaphysics. Even Hankins has some difficulty in bringing clarity to this subject, so some points will be highlighted. Hamilton created a triad of the mental faculties: reason, judgment and understanding. From these, he created a trinary tree, subdividing each one until he obtained twenty-seven categories. In October of 1840, John Graves asked Hamilton to abandon metaphysics and concentrate more on the mathematics. This only drove Hamilton to work more on metaphysics. Hankins states that William was very depressed from 1839 to 1842 because his wife, Helen, was living in England with her sister and because his close cousin Arthur died in 1840 (284-288). Whatever were the reasons for Hamilton to pursue metaphysics so heavily; he must have felt that it was the path to solving triplets.

DISCOVERY

On October 16, 1843, Hamilton discovered the key to solving his algebra. His answer, as previously stated, was to turn the triplets into quaternions. Hankins does not know what specifically caused Hamilton to resume his work on triplets. But, he describes Hamilton's discovery while walking in the morning with Helen along the Royal Canal to Dublin. William had a circuit close in his mind and the spark of the equations of i , j and k came out (which were previously introduced). It occurred to him that since the right side of the triplets had four terms that perhaps the left side should have four terms instead of three. He then wrote out the equations in a notebook immediately and during the ride to the Academy in the same afternoon he quickly worked out the preliminary details for a presentation. Hankins says that in one of Hamilton's later documents, he claims to have scribbled the equations on Brougham Bridge as he crossed it, however Hankins believes this was more a figurative comment (293-300). On the other hand, O'Donnell believes that Hamilton actually made the inscription. Regardless, a commemorative plaque now marks the famous spot (137-138).

Hamilton's colleagues, John Graves and Augustus De Morgan, were quite surprised that the discovery was quaternions since William was working on triplets. Hankins says the following:

Neither of them had realized that one might construct new elements of algebra rather than find the from the elements of existing algebra.[...] Graves obviously had not thought it permissible for Hamilton to create quaternions.[...] De Morgan was also caught by surprise, but he immediately recognized the importance of Hamilton's approach.[...] The Principle of the Permanence of Form was shattered by Hamilton's discovery, and the road was open to a wide variety of algebras that did not follow the rules of ordinary arithmetic.

On the same day of the discovery, Hamilton conjectured a metaphysical representation for quaternions where the imaginary parts might represent three-dimensional space while the real part would represent time (300-301).

IMPACT

The quaternion discovery did not spur mathematicians to study them and find applications. Instead, according to Hankins, "... led to a flurry of activity among British mathematicians, who were now prepared to forsake the Principle of Permanence of Form and follow Hamilton in the search for new algebras." Some thought that maybe algebras of higher order were possible, but Hamilton thought that more rules of algebra would have to be disregarded to make it consistent. Later, he would be proven right (302).

In an attempt to find an application for quaternions, Hamilton tried viewing them geometrically with the real part as a scalar and the imaginary parts as a vector. But according to Hankins, ultimately this was unsatisfactory because it conflicted with his views on number couples where the imaginary parts were removed. However, physicists appreciated the simplicity of vectors (triplets) to represent lines even if the algebra had shortcomings; it only required changing the imaginary basis to a unit basis (315, 318). So, quaternions did make a contribution to modern vector analysis.

It took five years for Hamilton to write *Lectures on Quaternions* which was over 800 pages.¹ Part of the difficulty in reading *Lectures* is that the mathematics is sparsely interspersed between his philosophies on metaphysics. O'Donnell states that very few people were able to read the book and very few copies were ever sold. Hamilton realized that another simpler book was required to help promote quaternions, so about five years after *Lectures*, he obtained a grant to produce a 300 page book within two years. Unfortunately, Hamilton could not help himself: he was trying to write a piece of history that would rival Newton's *Principia* (178-180). In the end, Hamilton never finished *Elements of Quaternions*; it was published posthumously in two volumes (1899 and 1901) because it was well over 1100 pages long.

1 Officially, it is listed at 736 pages, but with the preface and table of contents, it was well over 850 pages.

CONCLUSION

William Hamilton was extremely gifted and perhaps this ultimately prevented him from achieving greater status in history. Perhaps he was so brilliant that, unlike most mathematicians, he had no problems working with quaternions. He had recognized the scalar and vector parts in the quaternion, but was unwilling to sacrifice the algebra to develop it into vector analysis that others clearly did. So, although quaternions was his greatest discovery, its limited practicality makes Hamilton's greatest contribution to mathematics as that of helping other mathematicians into a new way of thinking and helping them achieve a piece of history. Furthermore, because his life and work was well documented, a blueprint exists for future students to become great mathematicians.

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