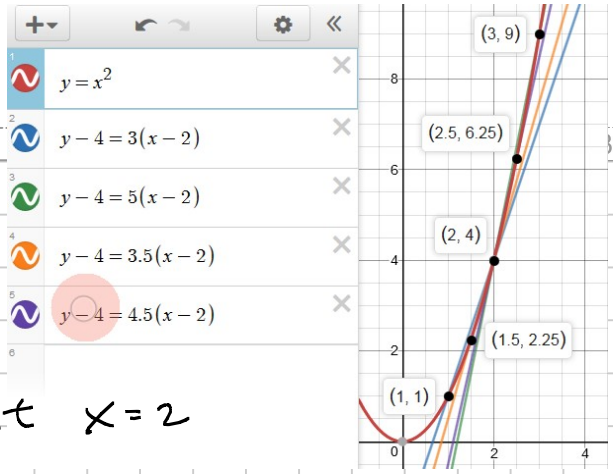
Calc 12 - Chp 2.1

Note Title

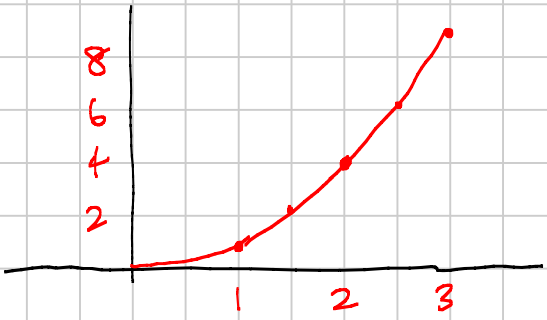
Limits Part 1

Estimating Tangent.

$f(x) = x^2$, find tangent at $x = 2$



x	f(x)
1	1
1.5	2.25
2	4
2.5	6.25
3	9



tangent using 1 & 2 :

tangent using 2 & 3 :

tangent using 1.5 & 2:

tangent using 2 & 2.5:

As we take smaller intervals, we get closer to the answer!

Limit Notation:

$$\lim_{x \to a} f(x) =$$

What we say: The limit as 'x' approaches 'a' of some function is the value 'L'.

Using limits: $\text{tangent} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ @ $x=2$

$h = x_2 - x$

Definition: Piecewise functions are multiple functions in a partitioned domain. (a domain that is divided up up into not necessarily even sized intervals).

eg) Absolute Value fn: $|x| = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$

Other uses for limits: Fill in holes for continuity using piecewise functions.

eg) $f(x) = \frac{x-1}{x^2-1}$, find $\lim_{x\to 1} f(x)$. Note that $f(1)$ is undefined.

NPV: $x = \pm 1$

Change $g(x) = \begin{cases} & x \neq 1 \\ & x = \pm 1 \end{cases}$

piecewise function →

$\lim_{x\to 1} g(x) = \frac{1}{2}$ still! $\neq g(1) = 2.5$

This is because the limit only considers the points near a value and not the value.

Properties of Limits:

$\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$

Limit Theorems: Given $L, M, c, k \in \mathbb{R}$ and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

Sum Rule: $\lim_{x\to c}[f(x)+g(x)] = L + M$

Difference Rule: $\lim_{x\to c}[f(x)-g(x)] = L - M$

Product Rule: $\lim_{x\to c}[f(x)\cdot g(x)] = L \cdot M$

Constant Multiplier Rule: $\lim_{x\to c} k\cdot f(x) = k\cdot L$

Quotient Rule: $\lim_{x\to c}[f(x)/g(x)] = L/M, \quad M \neq 0$

Power Rule: $r, s \in \mathbb{Z}$: $\lim_{x\to c}[f(x)]^{r/s} = L^{r/s}$

(direct substitution)

We can't use these rules on a polynomial directly, but we can use it on each term.

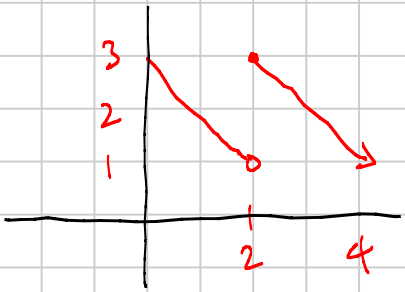
eg) $\lim_{h\to 0} \dfrac{(1+h)^3 - 1^3}{h} =$

One-sided Limits:

left-hand: $\lim_{x\to c^-} f(x) = L$ as x approaches 'c' from the left

right-hand: $\lim_{x\to c^+} f(x) = L$ ax x approaches 'c' from the right

eg)



$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2^+} f(x) =$

Two-sided Limit:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L$$

Usually apply to piecewise functions. For previous example, $\lim_{x \to 2} f(x)$

another eg) $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

Limit Theorems.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Direct Substitution - can only use when the fn is continuous (later) and you are told that you are allowed.

HW: pp. 62-64: 1-43 (odds), 32
Challenge: 45, 47

Calc 12 - Chp 2.2

Note Title 2013-09-23

Limits Part 2

Definition: Let 'c' and 'L' be real numbers. The function has limit 'L' as 'x' approaches 'c' if, given any positive number 'δ' (delta) such that for all 'x',

$$0 < |x-c| < \delta \Rightarrow |f(x)-L| < \varepsilon \text{ (epsilon)}$$

We write:

$$\lim_{x \to c} f(x) = L$$

eg) Prove: $\lim_{x \to -2} (2x+1) = -3$

Find $|x-(-2)| < \delta$ when $|(2x+1)+3| < \varepsilon$

$|2x+4| < \varepsilon$ simplify

$2|x+2| < \varepsilon$ factor

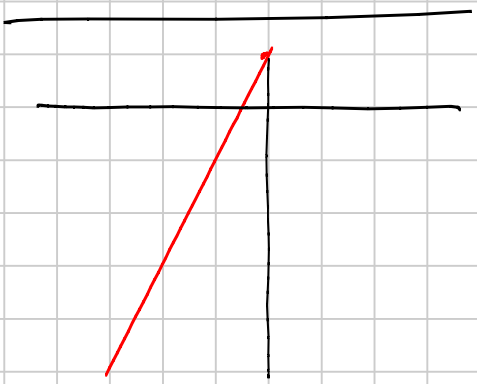
$|x-(-2)| < \varepsilon/2$ Suggest $\delta = \varepsilon/2$

Proof: (Use definition) Let $\varepsilon > 0$ be given, then

$$|(2x+1)-(-3)|$$
$$= |2x+4|$$
$$2|x+2| < 2\delta$$
$$< 2(\varepsilon/2)$$
$$< \varepsilon$$

If we take $\delta = \varepsilon/2$ and $0 < |x-(-2)| < \delta$ then we have $\lim_{x \to -2} (2x+1) = -3$

Graphically:



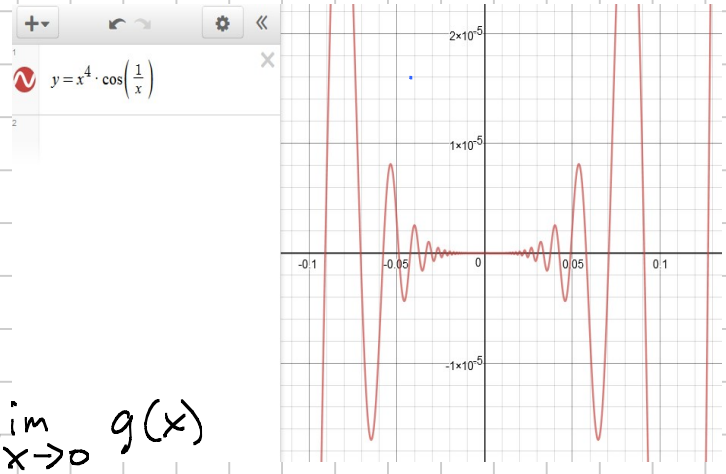
Sandwich Theorem: (AKA Squeeze Theorem)

If $f(x) \leq g(x) \leq h(x)$ $\forall x \neq c$ (for all) in some interval about 'c', and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} g(x) = L$$



eg) $g(x) = x^4 \cos\left(\frac{1}{x}\right)$, find $\lim_{x \to 0} g(x)$

Theorem: If $r > 0$ is a rational number ($\mathbb{Q}$), then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

Horizontal Asymptotes:

The line $y = L$ is a H.A. of the curve $f(x)$ if either

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L$$

eg) $f(x) = 1/x$

Left & Right H.A.'s may differ!

Theorem: If $r > 0$ is a $\mathbb{Q}$ such that (s.t.) x^r is defined for all ($\forall$) x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Limit Rules apply for $c = \pm\infty$.

eg) $\lim_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Vertical Asymptotes:

The line $x = c$ is a V.A. of the curve $f(x)$ if either

$$\lim_{x \to c^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = \pm\infty$$

eg) Find the V.A. of $f(x) = \dfrac{\sqrt{2x^2 + 1}}{3x - 5}$ Use NPV's

Finding End Behaviour Models

Definitions:

Left-end behaviour model for 'f' $\Leftrightarrow \lim_{x \to -\infty} \dfrac{f(x)}{g(x)} = 1$

Right-end behaviour model for 'f' $\Leftrightarrow \lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$

Left and Right may be different functions.

eg) Polynomial Rational functions are straight-forward.

$$f(x) = \frac{8x^6 - 3x^5 + 14}{-4x^2 + 13}$$

take highest power terms in numerator and denominator since highest terms dominate.

eg) $f(x) = e^x - x$

Compare x^n with n^x on graphing calculator

int fn.

Exponentials grow faster than polynomials

Hw: p. 64: 53 & 55
pp. 71-72: 1-13 (odds), 17, 19, 23, 29-37 (odds), 39, 32
Challenge: 15, 21, 25, 27, 39, 41
Use limits def to prove $\lim_{x \to 6} (\frac{x}{3} + 2) = 4$

Calc 12 - Chp 2.3a

Note Title 2013-09-23

Continuity - Big idea is that if we can draw a fn without lifting our pen, then the fn is continuous. However in math we need to show this with limits and theorems.

Definition: A fn 'f' is continuous at an interior point 'a' of its domain if

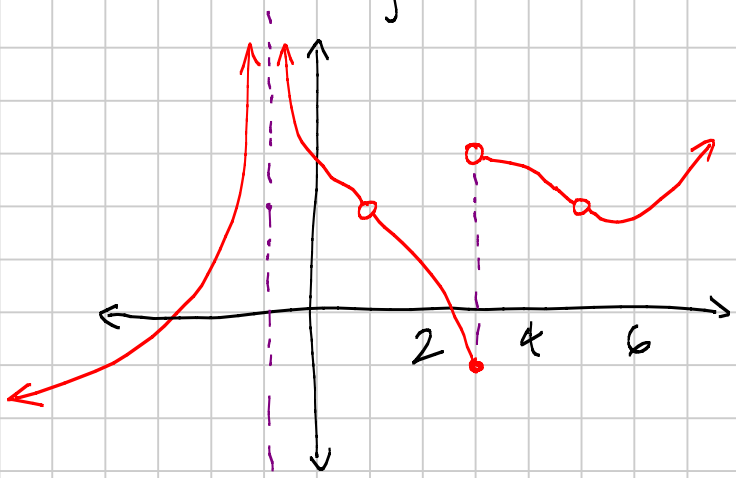
If 'f' is not continuous at 'a', then we say 'f' is at 'a' and 'a' is a

Definition: A fn 'f' is continuous at a left endpoint 'a' of its domain if

Definition: A fn 'f' is continuous at a right endpoint 'a' of its domain if

Types of Discontinuity

eg)



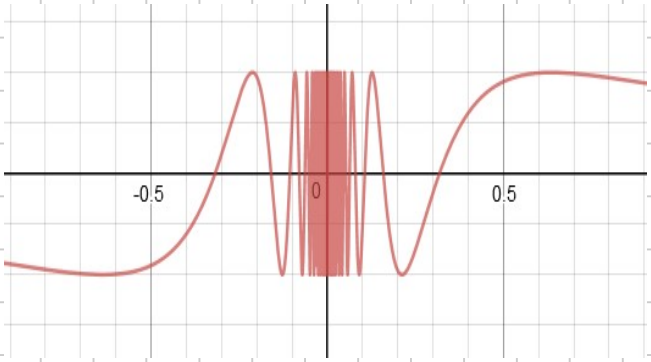
Point discontinuity is
We can remove the discontinuity with a piecewise fn.

eg) $f(x) = \frac{x^2 - x - 2}{x - 2}$. Remove the discontinuity.

discontinuity is at NPV, so $x = 2$

Oscillating Discontinuity

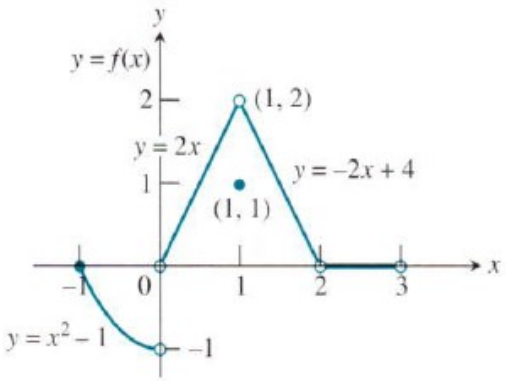
eg) $y = \sin(1/x)$



e g)

In Exercises 11–18, use the function f defined and graphed below to answer the questions.

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$



12. (a) Does $f(1)$ exist?

(b) Does $\lim_{x \to 1} f(x)$ exist?

(c) Does $\lim_{x \to 1} f(x) = f(1)$?

(d) Is f continuous at $x = 1$?

Graphically

a)

b)

c)

d)

Algebraically

a)

b)

c)

d)

HW: pp. 80-81 : 1 - 27 (odds), 24

Challenge: 31, 33

Calc 12 - Chp 2.3b

Note Title 2013-09-23

Continuous Functions.

Definition: A fn 'f' is continuous on an interval if and only if ( ) it is continuous at every point in the interval.

Definition: A continuous fn 'f' is one that is continuous at every point in its domain.

eg) $y = 1/x^2$ is a continuous fn because it is cont. on its domain: $|x| > 0$, even though it is not cont. on $[-2, 1]$

Theorem: Any polynomial is cont. on

Theorem: Any rational fn is cont on its domain, ie. anywhere except at

Theorem: The following fns are cont. in their

root (odd) -
root (even) -
trig - cos & sin :
tan :

inv trig - arccos :
arcsin :
arctan :

exp fns -
log fns -
abs val fns -

Properties of Continuous Fns

If 'f' and 'g' are continuous at 'a' and 'c' is a constant, then the following fns are also cont. at 'a':

Proofs: We only modify one side of the equation and show that it can be modified to equal the other side. If stuck, usually use reverse engineering to figure out steps.

Proof of 'f+g':

Prove 'f·g':

Since 'f' and 'g' are continuous:

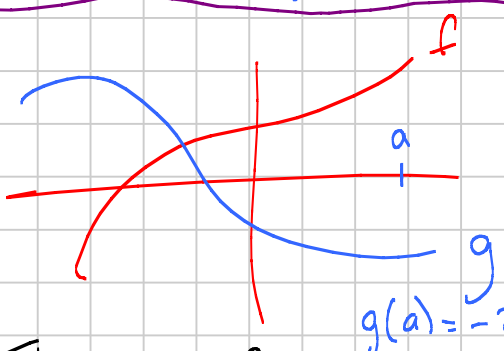
$$\lim_{x\to a} f(x) = f(a) \quad \& \quad \lim_{x\to a} g(x) = g(a)$$

$$\therefore \lim_{x\to a} (f\cdot g)(x) = \lim_{x\to a} \{f(x)\cdot g(x)\}$$

$$= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$= f(a)\cdot g(a)$$

$$= (f\cdot g)(a) \qquad \text{Q.E.D.}$$

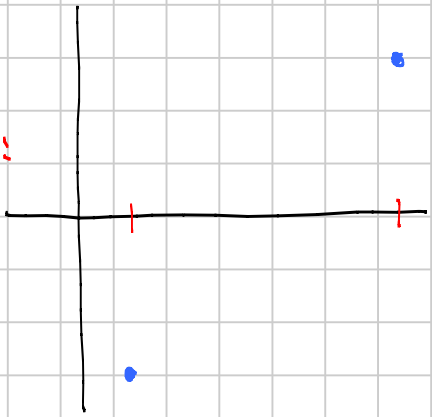


Theorem: Composition of Cont. Fns. If 'g' is cont. at 'a' and 'f' is cont at g(a), then 'fog' is cont. at 'a'.

Intermediate Value Theorem (IVT): Suppose that 'f' is cont. on an interval $[a,b]$ and let N be any number between $f(a)$ & $f(b)$ where $f(a)\neq f(b)$. Then there exists a number $c \in (a,b)$ s.t. $f(c) = N$.

Big Picture:

Why is there a condition:
$f(a) \neq f(b)$.



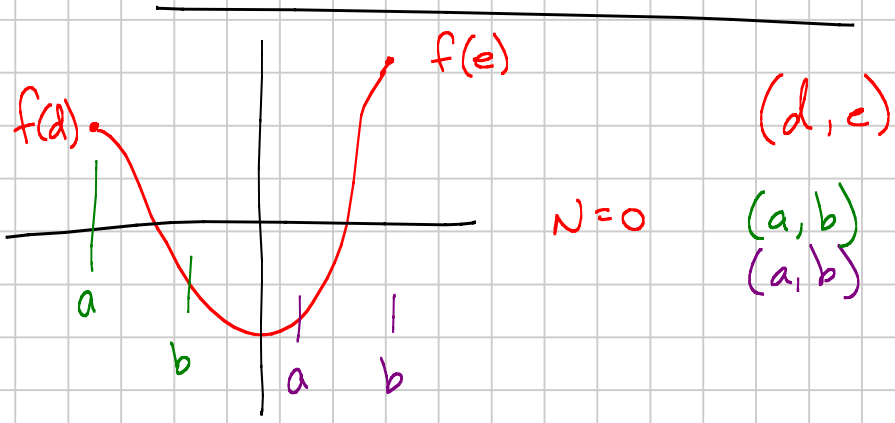
eg) Prove that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 3$$

between 1 & 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$

Consider:



$N = 0$

(d, e)

(a, b)

(a, b)

HW: p.81: 35-43 odds, 40

Challenge: 47, 49

hint: proof by contradiction

Consider to opposite, show this leads to a false statement. Then the original statement must be true.

eg) Show that there are an infinite number of whole numbers.

Assume that there aren't an infinite number of "

eg) Show $\sqrt{2}$ is irrational

assume $\sqrt{2}$ is rational

$a/b = \sqrt{2}$; $a, b \in \mathbb{Z}$; in lowest terms.

$a^2/b^2 = 2$

$a^2 = 2b^2$

'a' must be even

$\therefore$ b is odd b/c lowest terms.

$a = 2c$

$(2c)^2 = 2b^2$

$4c^2 = 2b^2$

$2c^2 = b^2$

'b' must be even

$\therefore$ 'b' must be even and odd

contradiction so $\sqrt{2}$ must be irrational.

Q.E.D.

---

eg) Show that $\sqrt{7}$ exists.

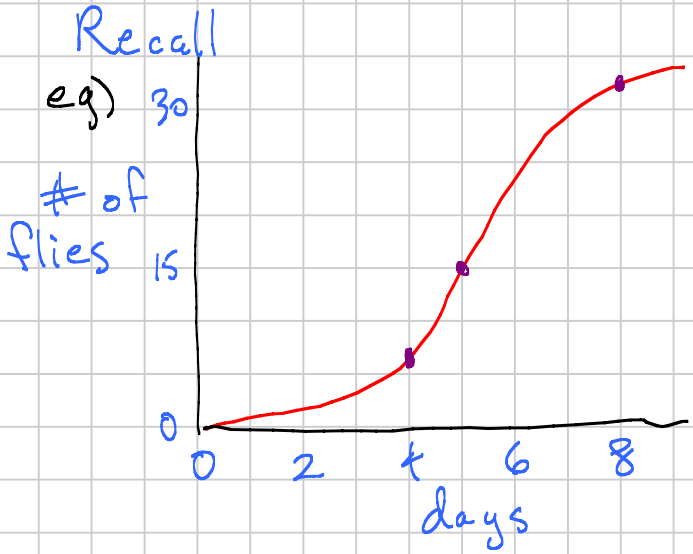
$x = \sqrt{7}$

We can't use square root as a fn, because that it is what we are trying to prove, so we must use a polynomial.

Calc 12 - Chp 2.4

Note Title 2013-09-23

Rates of Change and Tangent Lines



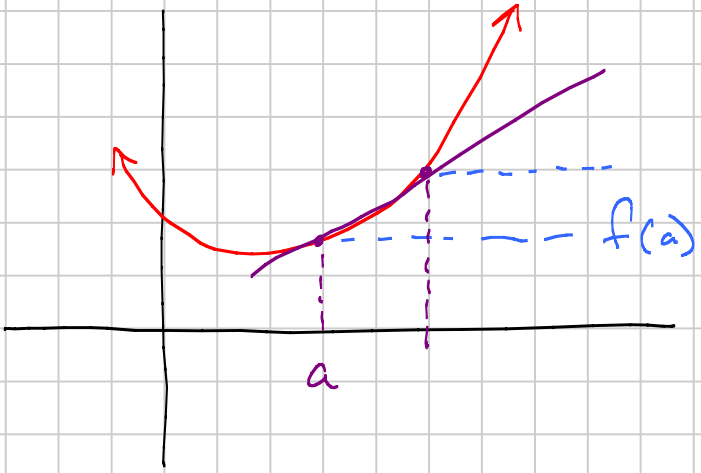
Average Rate of Change

$$\frac{\Delta f}{\Delta d} = \frac{15-7}{5-4} =$$

$$\frac{\Delta f}{\Delta d} = \frac{32-15}{8-5} =$$

Slope of a Curve at $y = f(x)$ at the point $P(a, f(a))$ is the number:

provided the limit exists.

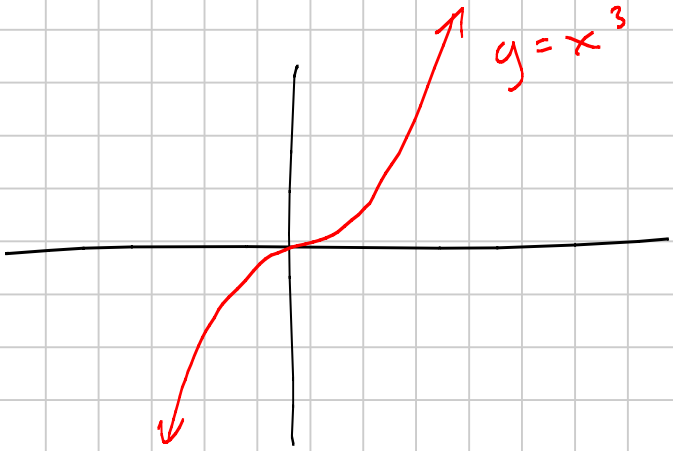


Tangent Line with above given then the line is

$$y - f(a) = m(x - a)$$

Normal to the Curve - it is $\perp$ to the tangent line to the curve.

eg) Let $f(x) = x^3$
Find slope at $x = a$.
When does slope $= 2$.
Find the tangent and normal lines at 'a'



T line:

N line:

eg) Let $f(x) = x^2 + 3$
Find normal and tangent lines at $x = 3$.

In Exercises 1–6, find the average rate of change of the function over each interval.

4. $f(x) = \ln x$

(a) $[1, 4]$ **(b)** $[100, 103]$

HW: pp. 87-89: 1–15, 19, 29, 14
Challenge: 23, 25