Calc 12 - Chp 8.2

Relative Rates of Growth

Definitions: for $f(x)$ & $g(x)$ positive & x sufficiently large

1. f grows faster than g (implies g grows slower than f) as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x \to \infty} \frac{g(x)}{f(x)} = 0$$

2. f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad L \neq 0 \text{ and finite}$$

You should notice that L'Hopital's Rule might come handy for this section.

eg) Show that e^x grows faster than x^2

$$\lim_{x \to \infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty}\ L'H}{\Rightarrow} \lim_{x \to \infty} \frac{2}{e^x} = 0$$

so e^x grows faster than x^2

eg) Show that x grows faster than $\ln x$.

$$\lim_{x \to \infty} \frac{x}{\ln x} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{x \to \infty} \frac{1}{1/x} = \lim_{x \to \infty} x = \infty$$

so x grows faster than $\ln x$.

Transitivity of Growth Rates.

If f grows at the same rate as g as $x \to \infty$ and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h.

If $\lim_{x \to \infty} \frac{f}{g} = L$ and $\lim_{x \to \infty} \frac{g}{h} = M$ then $\lim_{x \to \infty} \frac{f}{h} = LM$

eg) Show $\sqrt{2x^3-5}$ and $(3\sqrt{x}+1)^3$ have the same growth rate.

We can use L'Hopital's Rule or just use transitivity.

$f(x) = \sqrt{2x^3-5}$ $\quad h(x) = (3\sqrt{x}+1)^3$

Pick an easy $g(x)$. $\quad g(x) = x^{3/2} = \sqrt{x^3}$

$$\lim_{x\to\infty} \frac{\sqrt{2x^3-5}}{\sqrt{x^3}} = \lim_{x\to\infty} \sqrt{2-\frac{5}{x^3}} = \sqrt{2}$$

$$\lim_{x\to\infty} \frac{x^{3/2}}{27x^{3/2}+27x+9\sqrt{x}+1} = \lim_{x\to\infty} \frac{1}{27+\frac{27}{\sqrt{x}}+\frac{9}{x}+\frac{1}{x^{3/2}}} = \frac{1}{27}$$

expand with binomial theorem

By transitivity: $\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \left(\frac{f}{g}\,\frac{g}{h}\right) = \sqrt{2}\left(\frac{1}{27}\right) = \frac{\sqrt{2}}{27}$

Definition: f of Smaller Order than g, if

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then we write $f = o(g)$

and say "f is little-oh of g"

Definition: f of at Most the Order of g, if

$\lim_{x\to\infty} \frac{f(x)}{g(x)} \le M$, then we write $f = O(g)$

and say "f is big-oh of g"

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eg) Show $4\ln x = o(x^2)$

$$\lim_{x\to\infty} \frac{4\ln x}{x^2} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{x\to\infty} \frac{4(1/x)}{2x} = \lim_{x\to\infty} \frac{2}{x^2} = 0$$

eg) Show $2x + \cos x = O(x)$

for $x \ge 1$ because $2x + \cos x > 0$

$$\lim_{x\to\infty} \frac{2x+\cos x}{x} = \frac{2-\sin x}{1} \le 2+1 = 3$$

HW: p. 431: 1-23 odds (please show 20)