Counting and Probability is a field of mathematics called Combinatorics. We will study combinations, permutations, Pascal’s Triangle, and the Binomial Theorem. Counting is much harder than most people think!

The Fundamental Counting Principle - If there are \( n_1 \) different objects in one set and \( n_2 \) different objects in a second set, then the number of ways of choosing one object from each set is \( n_1 \cdot n_2 \). This can be applied to multiple sets: \( n_1 \cdot (n_2) \cdot (n_3) \cdot (n_4) \cdots \)

This applies to what we call each event has no impact on the next event.

eg) How many ways can we pick one card from each suit in a regular deck of cards? Each suit is independent of the others.

eg) How many ways can we pick one card from reds and one card from blacks in a regular deck?

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Tree Diagram - this can be used for small cases. Draw all nodes then count the leaves. Trees are drawn upside-down. This generates all the possible outcomes.

eg) Draw a tree for flipping a coin 3 times.
Pigeonhole Principle - If there are $n$ items and $m$ pigeonholes and $n > m$, then at least one pigeonhole contains more than one item.

**Example**

How many regular dice must be thrown to ensure a match?

**Example**

How many combination locks can the lock company make before there is a duplicate combination. The locks use 3 numbers from 1 to 30.

Backwards problems.

**Example**

A slot machine has odds of 10,648 to 1 of getting a jackpot. If there are 3 reels on the machine, how many stops are on each reel?

**Example**

A slot machine has odds of 53,782 to 1 of getting a jackpot. There are 14 stops on each reel. How many reels are there? Harder
A restaurant offers combo meals, you can choose:
- a cheeseburger, chicken strips, or cheese pizza.
- fries, or onion rings
- pop, milk, coffee, tea, or juice
- ice cream, pie, or cake

How many different meals are there?

A partial sample space:

Non-repetition: These problems are slightly different because the number of choices won't stay fixed.

e.g.) How many ways can 3 cards be dealt from a deck of cards?

Don't use formulas blindly! Determine the number of choices such as dependence. When ensuring a match, it is likely that it is the Pigeonhole Principle.

Note: A deck of cards has
- 52 cards
- 26 black & 26 red
- 13 spades (black), 13 hearts (red), 13 diamonds (red), 13 clubs (black)
- Ace, 2-10, Jack, Queen, King of each suit

HW: pp. 689-693: 3-10, 12-14
Challenge: 15, 16
3 not exactly PCF.
Please solve 11 in person
Permutations of Different Objects.
To permute is to arrange. Permutation is to find all the different ways to arrange objects, so order is important. This is a slight variation on FCP. We are arranging objects that are NOT identical.

*eg*) How many ways can HAT be arranged?

We can generalize for any $n$ objects that the number of arrangements is: $(n)(n-1)(n-2)\cdots(3)(2)(1)$. There is a function for this operation called factorial. The notation is $n!$. By definition,

*eg*) How many ways can LUCKY be arranged?

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MATH NUM CPX RES
1:rand
2:nPr
3:nCr
4:1
5:randInt()
6:randNorm()
?randBin()
```

$5! = 120$
If only \( r \) objects is to be selected from a set of \( n \), then the formula is:
\[
(n)(n-1)(n-2) \cdots (n-r+2)(n-r+1) = nPr = \frac{n!}{(n-r)!}, \quad n \geq r
\]

\[\text{eg)}\]
How many ways can OBJECT be arranged when choosing 3 letters?

\[\text{eg)}\]
There are 9 teams in the Western Conference of MLS. How many ways can the playoff standings be arranged if 5 teams make the playoffs?

\[\text{eg)}\]
There are 12 horses in a race. How many ways can they finish first, second, and third (win, place, and show)?

Arrangements of Arrangements. Sometimes you will need to keep some items together.

\[\text{eg)}\]
4 families are sitting in a row of seats. The families have 5, 4, 3, and 4 members each. Each family needs to sit together. How many arrangements are there?
If the families didn’t have to sit together, there would be $16! = 2.09 \times 10^{13}$ ways which is a far larger number.

The backwards problem.

eg) Solve for $n$: $n \cdot P_3 = 210$

If you are stuck, use trial and error:

$10 \cdot P_3 = 720$ too high.
$5 \cdot P_3 = 60$ too low.
$8 \cdot P_3 = 336$ too high.
$7 \cdot P_3 = 210$ got it!

eg) Harder: Solve: $13 \cdot P_r = 17160$ Use guess and test.

HW: pp. 702-705: 3 - 7, 9, 10
Challenge: 11, 12
Please solve 8 in person.
Permutations Involving Identical Objects.

eg) How many ways can CANADA be arranged?

\[ CA_1NA_2DA_3 = CA_2NA_1DA_3 = CA_3NA_2DA_1 \]

There are 3! ways to arrange A_1A_2A_3.

eg) How many ways can MISSISSAUGA be arranged?

eg) How many ways can 1222566 be arranged?

eg) How many ways can 444444 be arranged?

eg) A 34 term series consists of -4 and +13. How many ways can they be arranged to a sum of 0?

For problems that aren't written as letters or digits, we can use them in place of objects to conceptualize for an easy solution.
eg) How many ways can SEA BISCUIT be arranged keeping the vowels together?

Backwards Problem

eg) A word that contains 5 letters and has 20 arrangements

If $x = 2^4$, this could be $4!$ or $3! \cdot 2! \cdot 2!$

but $x = 6$ or $x = 12$ are not ambiguous.

Again, don’t just use formulas, take time to consider if it is FCP, arrangements, or arrangements with duplicate objects.

HW: pp. 712-715: 4-10, 12-13
Challenge: 14
Please solve 11 in person
Combinations - used when order does not matter.
\[
\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!} = \left( \binom{n}{r} \right)
\]

This is similar to the treatment of identical objects in an arrangement.

Eg) Wendy’s has 8 items on their Value Menu. How many ways are there to order 3 items? Does it matter if we order a Frosty, Jr. Bacon Cheeseburger, and fries versus ordering fries, Jr. Bacon Cheeseburger, and Frosty?

Eg) 10 students are running for student council. 6 will get elected. How many ways can the council get formed?

You must be able to determine whether to use permutations or combinations because you won’t always be told.
There is symmetry in combinations! but not for permutations.

\[ \binom{n}{r} = \binom{n}{n-r} \]

\[ \binom{6}{2} = \binom{6}{4} \quad n \binom{3}{3} = n \binom{3}{0} \]

\[ LHS = \frac{n!}{r!(n-r)!} \]

\[ = \frac{n!}{r!(n-r)!} \quad r = n - (n-r) \]

\[ RHS = \frac{n!}{(n-(n-r))!(n-r)!} \]

\[ Q.E.D. \]

\[ \text{eq)} \quad \text{Given that Lotto 6/49 has odds } 49 \binom{6}{0} = 13983816 \] to 1 of winning the jackpot, what are the odds of winning the Lotto 49/49 jackpot, where you need to pick 49 numbers?

If there are multiple combinations, we use the Fundamental Counting Principle and multiply the answers together.

\[ \text{eq)} \quad \text{An ice cream store has 10 flavours of ice cream and 21 mix-in flavours. How many flavours can you get if you are allowed 2 ice cream choices and 2 mix-in choices?} \]
Backwards Problem

e.g.) Find \( n \) or \( r \).

\[ \begin{align*}
\binom{n}{2} &= 36 \\
\binom{10}{r} &= 120
\end{align*} \]

Again, you want to think about what situation applies: does arrangement matter, if not then it is like a combination problem.

HW: pp. 727-732: 4-13, 15
Challenge: 16, 17
Please solve 14 in person
Pascal's Triangle

Row

1
2
3
4
5
6
7
8
9

Notice patterns.

There is a relation between Pascal's Triangle and Combinations:

\[ \binom{n}{r} \text{ has the same value as the } r^{th} \text{ term in that row.} \]

Use Combinations instead of Pascal's Triangle for large or individual numbers to avoid computing a large triangle.

Eg) Find the 8th and 10th terms in the 12th row in Pascal's Triangle.

HW: pp. 735-737: A-6, 1-4
Please solve "the fifth term in row 23" in person
The Binomial Theorem

Pascal's Triangle - Identify Features:

\( n \text{C} r \) - row number is \( n+1 \)
\( n+1 \) is the number of terms in the row
\( n \) is the second and second last term in the row.

\[
\begin{align*}
(x+y)^0 &= 1 \\
(x+y)^1 &= 1x + 1y \\
(x+y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3 \\
(x+y)^4 &= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \\
&\vdots
\end{align*}
\]

This gives us the Binomial Theorem:

\[
(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n
\]

\[
(x+y)^n = x^n + nx^{n-1}y + n\binom{n-1}{1}x^{n-2}y^2 + n\binom{n-1}{2}x^{n-3}y^3 + \ldots + nxy^{n-1} + y^n
\]

The \( k^{th} \) term of \((x+y)^n\) is:

\[
e g\) Substitution for \((3a-2b)^4\) \quad x = 3a \quad y = -2b
\]

\[
= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4
\]
Determine the 4th term of \((x^2 - 4x + 4)^5\).

Determine the binomial power when given:
\[
81x^4 - 54x^3y + 27x^2y^2 - \frac{3}{2}xy^3 + \frac{y^4}{16}
\]

Determine the binomial power when given:
\[
\frac{1}{27}x^3 - \frac{4}{3}x^2y + 16xy^2 - 64y^3
\]

HW: pp. 743 - 749: 3a, 4, 5, 7bd, 8, 9, 13, 15, 17
Challenge: 16, 18
Please solve 14 in person.