Classifying Conics: \( x^2 + y^2 = z^2 \)

Conic sections are formed by intersecting the cones with a plane of the equation:
\[ ax + by + cz + d = 0 \]

We can simplify by making \( a = 0 \).
\( \beta \) is the angle of the cone slope. For simplicity, we will keep it \( 45^\circ \).

We can create circles by using a horizontal plane to create our intersection:
\( (x = 0^\circ) \)
\[ cz + d = 0 \]

We can create parabolas by using a plane parallel with the edge of the cone:
\( (x = 45^\circ) \)
\[ z = -by - d, \quad d \neq 0 \]

We can create ellipses by using a plane with an angle to the horizontal s.t. \( (x < 45^\circ) \)
\[ z = -by - d, \quad 0 < b < 1, \quad d \neq 0 \]
Finally, we can create hyperbolas by using a plane with an angle to the horizontal such that: $\alpha > 45^\circ$.

$\beta = -by - d$, $b > 1$, $d \neq 0$

With $\alpha = 0$, we can calculate: $\alpha = \tan^{-1}(\beta/c)$

The above equations are not the typical forms.

Circle: $x^2 + y^2 = r^2$

Parabola: $y^2 = 4ax$ or $x^2 = 4ay$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

All conic equations are just a special form of the general form: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

From the general form, we can determine the type of conic from $B^2 - 4AC$. Assuming conic is not degenerate (lines/point):

- $< 0$: parabola
- $= 0$: line
- $> 0$: ellipse

Notice that all the conics, except for vertically opening parabolas, are not functions; they are relations.

Definition: Locus - a set of all points that satisfy a distance relationship: the red lines.
Conics have properties.

The orange dot is the focal point (parabolas don’t have one).
The black dots are the foci (plural of focus).
Focal length is the eccentricity (another way to determine the type of conic).

Conics have circles and
Conics have circles and parabolas have
Ellipses and hyperbolas have
e and

e.g. Classify:
\[ x^2 + y^2 = 49 \]
\[ \frac{x^2}{4} + \frac{y^2}{81} = 1 \]
\[ 4(x-2) = y^2 \]
\[ \frac{y^2}{81} - \frac{x^2}{25} = 1 \]

e.g. Classify and write in standard form.
\[ 3y^2 + 10x - 6y + 25 = 0 \]
\[ x^2 + y^2 + 6x - 8y - 50 = 0 \]
\[ x^2 - y^2 + 4y - 6 = 0 \]
\[ 3x^2 + 5y^2 + 12x - 30y - 3 = 0 \]

HW: Classify Conics: 1-15 odds
Properties of Circles
Even though we are already familiar with parabolas, we start with circles because they are the simplest.
You will need to complete the square for \( x \) and \( y \) to find the properties.

Standard form:
- **Vertex:** \((h, k)\)
- **Radius:** \( r \), \( r > 0 \)
- **Center and Focus:** \((h, k)\)
- **Directrix:** line @ \( x = \infty \)
- **Axis of Symmetry:** every diameter
- **Eccentricity:** 0
- **Latus rectum:** every diameter
- **Translation from origin:** \( h < 0 \), left; \( h > 0 \), right
- \( k < 0 \), down; \( k > 0 \), up

Additional information to recall to determine properties:
- **Pythagorean:** 
  \((x - h)^2 + (y - k)^2 = r^2\)
- **Tangent line:** this line will only touch one point of the circle that is 90° to the radius at that point.

**Area of circle:** \( \pi r^2 \)

**Circumference:** \( 2\pi r \)

**Graphing:** Plot the center, then use the radius.

**Example:**
5) \( x^2 + y^2 + 4x + 2y - 4 = 0 \)

11) Center lies in the second quadrant
Tangent to \( x = -14 \), \( y = -4 \), and \( x = 8 \)
10) \( x^2 + y^2 + 24x + 10y + 160 = 0 \)

Identify \((h, k)\) & \(r\)

Find the equations in standard form:

10) \((x + 5)^2 + (y + 7)^2 = 36\)
Translated 5 left, 4 down

26) Three points on the circle:
\((-7, 6), (9, 6), \) and \((-4, 13)\)

HW: Circles: 2-14 every other even
Graphing circles: 1-13 every other odd
Equations of circles: 1-29 every other odd
Properties of Parabolas

You will need to complete the square to find most properties. If you can find the x-intercepts, you can use the average to find the axis of symmetry then the vertex.

Once you find the vertex, you need the standard form:

\[ 4a(y-k) = (x-h)^2 \quad \text{or} \quad 4a(x-h) = (y-k)^2 \]

New parabola definition: is a locus where the distance to the focus is equal to the distance to the directrix.

The axis of symmetry passes through the focus and vertex and is perpendicular to the directrix.

Eccentricity:

We can calculate the focus:

or

The directrix is:

\[ y = k - \frac{a}{e} = k - \frac{a^2}{c} \quad \text{or} \quad x = h - \frac{a}{e} = h - \frac{a^2}{c} \]

The goes through the focus and is perpendicular to the axis of symmetry until it touches the parabola.

The length is \(|4a| = |4c|\)

In addition to properties, notice that parallel lines coming into a parabola intersect at the focus. Revolving a parabola about the axis of symmetry, we can create a parabolic reflector, paraboloid. A paraboloid can turn weak signals into a strong one (energy, sound, electro-magnetic waves). Examples are reflecting telescopes, satellite dishes, microphones, flashlights, and solar power.
3) \(-\frac{1}{3}(x - 3) = (y + 5)^2\) 

Graph and find properties

Find the equation in standard form:

9) Vertex: \((-1, -3)\), Focus: \(\left(\frac{17}{16}, -3\right)\)

12) Opens left or right
   Vertex: \((-7, 9)\)
   Passes through: \((-4, 8)\)

22) Vertex: \((4, 2)\), axis of symmetry: \(x = 4\),
    length of latus rectum = \(\frac{1}{3}\), \(a > 0\)

HW: Parabolas: 2-14 every other even
Graphing Parabolas: 6, 10, 14, 17, 18
Equations of Parabolas: 1, 3, 5, 7, 19, 21
Properties of Ellipses
Like circles, you will need to complete the square for \(x\) and \(y\) to find the properties. \(a > b\)

Standard form:

\[c = \sqrt{a^2 - b^2}\]

Center: \((h, k)\)

Vertex: \((h \pm a, k)\) \((h, k \pm a)\)

Foci: \((h \pm c, k)\) \((h, k \pm c)\)

Major Axis: \(y = k\)

Minor Axis: \(x = h\)

Eccentricity: \(0 < \frac{c}{a} < 1\)

Directrices:

\[x = h \pm \frac{a^2}{c}\]

\[y = k \pm \frac{a^2}{c}\]

Length of Latus Rectum: \(\frac{2b^2}{a}\)

Determine the properties and graph:

2) \(\frac{(x - 3)^2}{5} + \frac{(y - 1)^2}{15} = 1\)
8) $36x^2 + 5y^2 - 90y - 495 = 0$

Find the equation in standard form:

10) Vertices: $(13, 9), (-3, 9)$
   Foci: $(5 + 2\sqrt{7}, 9), (5 - 2\sqrt{7}, 9)$

12) Foci: $(3, 10 + \sqrt{105}), (3, 10 - \sqrt{105})$
   Co-vertices: $(11, 10), (-5, 10)$

16) Eccentricity $= \frac{\sqrt{15}}{4}$
   Center: $(-5, 5)$
   Co-vertex: $(-8, 5)$

HW: Ellipses: 1, 3, 5, 7, 9, 11, 13, 15
Graphing Ellipses: 6, 13, 18
Equations of Ellipses: 11, 13, 15, 20, 26
Properties of Hyperbolas

Like the other conics, you will need to complete the square to get the hyperbola into standard form.

Standard form: horizontal

$$C = \sqrt{a^2 + b^2}$$

Center: $$(h, k)$$  The difference of distances from focus to foci is constant

Vertices: $$(h \pm a, k)$$  $$y = k$$

Foci: $$(h \pm c, k)$$  $$x = h$$

Major Axis:  $$X = h$$

Minor Axis:  $$y = k$$

Eccentricity: $$e = c/a > 1$$

Directrices: $$x = h \pm \frac{a^2}{c} = h \pm \frac{a}{e}$$

Length of Latus Rectum: $$2\frac{b^2}{a}$$

These properties are the same as the ellipse! New:

Slant asymptotes: $$y = \pm \frac{b}{a}(x-h)+k$$

Use the slant asymptotes to help you draw the hyperbolas more accurately.

Use ellipse strategies to complete problems.

The central rectangle is simply extending the major and minor axes until they form corners on the slant asymptotes.
2) \((y + 4)^2 - (x - 3)^2 = 1\)

Find the properties and graph:

6) \(-9x^2 - 32y = -16y^2 + 128\)

Find the equation in standard form:

13) Center at \((10, -4)\)
   - Transverse axis is vertical and 18 units long
   - Conjugate axis is 10 units long

5)
9) Vertices: (15, 1), (−1, 1)
   Endpoints of Conjugate Axis: (7, 7)
   (7, −5)

15) Foci: (2, −6 + 15\sqrt{7}), (2, −6 − 15\sqrt{7})
   Asymptotes: \( y = \frac{6}{11}x - \frac{78}{11} \)
   \( y = \frac{6}{11}x - \frac{54}{11} \)

21) Foci: (−10, 10 + 4\sqrt{10}), (−10, 10 − 4\sqrt{10})
   Points on the hyperbola are 24 units closer to one focus than the other

HW: Hyperbolas: 1, 3, 5, 7, 10, 12, 16
   Graphing Hyperbolas: 1, 5, 8
   Equations of Hyperbolas: 6, 8, 16